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THE CONTROL OF THE ANGULAR MOTION OF A SOLID WITH INTERFERENCE. A GAME-THEORETIC APPROACH†

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A constructive method for control synthesis for the problem of the reorientation of an asymmetric solid is presented using a game-theoretic model of interference. Prescribed "geometric constraints" on the control functions are taken into account. The method is based on a choice of the structural form of control rules that enables the solution of the original non-linear problem to be reduced to solving auxiliary linear game-theoretic problems. The non-linear control rules obtained in this way are robust (stable) and ensure that the body arrives at the prescribed state in a finite time, which can be computed within the framework of the proposed solution scheme starting from the time-optimization requirement. A method of estimating the guaranteed minimum reorientation time is given. The results of a computer simulation are presented.

The possibility of extending the approach in question to the case of reorientation with simultaneous damping of the angular velocity of the body is considered.

From the standpoint of general problems of control theory the proposed approach is close to the methods of decomposition, partial stabilization, and exact linearization of non-linear control systems, which have been extensively developed in recent years.

There are now many publications devoted to the problems of controlling the angular motion of a solid (see, for example, [1–13] and the bibliography therein). Problems of this kind are of interest in the study of miscellaneous problems that arise, for example, in the dynamics of aircraft and spacecraft, robotics, and biomechanics. In a strictly non-linear formulation it is hard to find optimum solutions for such problems, and they are often divided into two stages: rotation damping and reorientation in space. At the first stage the study is based on Euler's dynamical equations, a knowledge of the angular position of the body being unnecessary. At the second stage the initial and final states are stationary states. The orientation can be altered in a number of ways [1–13]: (1) by three successive rotations about connected axes; (2) in the class of planar rotations about the Euler axes (extensive turn), in which case the angular vector of the body keeps a fixed position in space; (3) several planar rotations; (4) one spatial turn without any additional restrictions on the character of the resulting motion. The question of which of the schemes is appropriate for optimal control is solved separately in each case. As a rule, the prescribed speed and fuel consumption as well as reliability and technological requirements are taken into account.

The optimization problems of controlling the extensive turn are most completely reflected in the literature. This method is particularly effective in those cases when the inertia ellipsoid of the body is close to a sphere or the control is realized by small control moments [7]. Problems

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concerned with the optimum reorientation of a solid body by a single spatial rotation have been less extensively studied [11].

The actual conditions under which objects function make it necessary to take into account constantly acting interference and perturbations. The control process becomes especially difficult if only the minimum information about the interference, namely their limits of variation, is available. Here one is concerned with control under conditions of uncertainty. In this case, however, as is shown in this paper, for the stage of altering the orientation of the body one can put forward a constructive method of solution subject to the constraints on the control, which, even if not rigorous, is completely acceptable from the viewpoint of applications.

The proposed approach rests on choosing a structural form of control rules that enables one to reduce the original non-linear problem to auxiliary linear control problems. The additive "auxiliary interference" appearing in the auxiliary linear control systems (generated by an appropriate transformation of interference in the original system) are then treated as player–opponent control functions. As a result, the auxiliary linear control systems can be regarded as conflict-control systems. The parameters of the chosen structural form of the original control rules are determined as player–opponent control strategies in the corresponding linear game-theoretic problems. In this way one obtains robust (stable) non-linear control rules (as a synthesis) ensuring a guaranteed result when solving the original non-linear control problem. Namely, it is guaranteed that the solid will move precisely into the prescribed final state (by a single spatial rotation) in a finite time for any admissible form of interference. A constructive method of estimating the reorientation time is proposed, taking into account the given constraints on the control functions.

The proposed scheme for solving the problem is such that the reorientation time does not exceed the minimum guaranteed control time in the corresponding linear game-theoretic problem. In this sense the resulting rules for controlling the reorientation (making the most of the possibilities of the structural form proposed for them) can be called time-suboptimal with respect to the action from the given class.

The feasibility of extending the approach to the case of the problem of reorientation with simultaneous damping of the angular velocity of the body is discussed.

1. FORMULATION OF THE PROBLEM

Consider Euler's dynamical equations

$$A_1\dot{x}_1 = (A_2 - A_3)x_2x_3 + u_1 + \upsilon_1 \quad (1\ 2\ 3) \tag{1.1}$$

(one of the three equations is written down; the others can be obtained by a cyclic permutation of the indices $1 \to 2 \to 3$). Equations (1.1) describe the angular motion about the centre of mass of a solid subject to control moments u_i and moments υ_i characterizing the external forces and uncontrollable perturbations. Here and henceforth (unless otherwise stipulated) $i = 1, 2, 3$. In (1.1) x_i are the projections of the angular velocity vector of the body onto its principal axes of inertia and A_i are the main central moments of inertia of the solid. We denote by $\mathbf{x}, \mathbf{u}, \mathbf{v}$ the vectors formed by the components x_i, u_i, υ_i, respectively.

We shall consider Eqs (1.1) together with the kinematic equations

$$2\dot{\lambda}_0 = -(x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_3), \quad 2\dot{\lambda}_1 = x_1\lambda_0 + x_3\lambda_2 - x_2\lambda_3 \quad (1\ 2\ 3) \tag{1.2}$$

in terms of the Rodrigues–Hamilton variables which determine the orientation of the solid. In addition, the variables λ_i and λ_0 in the quaternion λ satisfy the relationship

$$\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1 \tag{1.3}$$

In a number of cases system (1.1)–(1.3) serves as a model of the angular motion of a spacecraft. The vector $\boldsymbol{\lambda}$ defines the direction (orientation) of the spacecraft relative to some physical frame of reference. The controlling moments u_i are realized by special engines and/or rotors (flywheels, gyroscopes).

The devices producing the controlling moments have limited capabilities. We shall assume that the "geometric constraints"

$$|u_i| \leqslant \alpha_i = \text{const} > 0 \tag{1.4}$$

are imposed on the controlling moments. Here α_i are given numbers characterizing the maximum possible magnitudes (levels) of u_i.

Various assumptions about the interference υ_i acting on the body can be made. If the various forms of interference are stochastic processes (for example, white noise), then a stochastic (probabilistic) approach to the solution is possible. As it applies to certain problems of controlling the angular motion of a solid (e.g. a spacecraft) such an approach has been considered, for example, in [14–17].

However, in many cases just the minimum information about the interference, namely its limits of variation, is available and no probabilistic characteristics of the various forms of interference within these limits are known. In this case only constraints of the form

$$|\upsilon_i(t)| \leqslant \beta_i = \text{const} > 0 \tag{1.5}$$

are imposed. The numbers β_i characterize the maximum possible interference magnitudes (levels), which can take the form of any piecewise continuous time-dependent functions $\upsilon_i(t)$ within the limits (1.5). As opposed to the stochastic approach, differential games [18–21] include control problems that guarantee the desired result even for the worst possible action of interference. This approach applied to the problems of controlling the angular motion of a solid is considered in the present paper.

Let us make the control problems under consideration more specific. Suppose that the class $K = \{\mathbf{u} : \mathbf{u} = \mathbf{u}(\mathbf{x}, \boldsymbol{\lambda}),\ \mathbf{x},\ \boldsymbol{\lambda}) \in D\}$ of piecewise-continuous functions (in an admissible domain D of $\mathbf{x}$ and $\boldsymbol{\lambda}$) that satisfy (1.4) is given, and also the class $K_1 = \{\boldsymbol{\nu} = \boldsymbol{\nu}(t)\}$ of interference that satisfies (1.5) and are piecewise-continuous in any finite interval $t \in [t_0,\ t_1]$.

Problem 1 (of reorientation). It is required to find control rules $\mathbf{u} \in K$ that will take the body from the initial state $\boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}^0$ to the prescribed state $\boldsymbol{\lambda}(t_1) = \boldsymbol{\lambda}^1$, where $(\boldsymbol{\lambda}^0,\ \boldsymbol{\lambda}^1) \in D$, in a finite time for any admissible form of interference $\boldsymbol{\nu} \in K_1$, Both states are stationary and $\mathbf{x}(t_0) = \mathbf{x}^0 = \mathbf{x}(t_1) = \mathbf{x}^1 = \mathbf{0}$. The time $t_1 > t_0$ is not fixed.

Problem 2 (of reorientation with angular velocity dampening). It is required to find a control rule $\mathbf{u} \in K$, which solves Problem 1 in the case when $\mathbf{x}^0 \neq 0$.

2. THE PROPOSED APPROACH TO THE SOLUTION

Because of the essential non-linearity of system (1.1)–(1.3) it is quite difficult to solve Problems 1 and 2 rigorously, especially when it is required in addition that the guaranteed reorientation time should be a minimum. This is also related to the constructive estimate of the guaranteed minimum reorientation time. In such a situation it is natural to seek simple control rules which, even if not time-optimal, enable one to obtain an acceptable reorientation time. Our aim is to find time-efficient control rules. Furthermore, we consider the solution of Problems 1 and 2.

To realize this aim we assume that the structural form (the construction) of the desired game strategies u_i is given. As a basis we take the constructions of control rules which solve the problems of controlling the angular motion of a solid such as those proposed in [22, 23] for the ideal case without interference ($\upsilon_i \equiv 0$). In this framework the original non-linear problem can be reduced to solving a number of control problems of one type for auxiliary linear systems. These systems have a simpler form $\ddot{\lambda}_i = u_i^*$ in the case of triaxial reorientation and $\ddot{s}_j = u_j^*$ ($j = 1, 3$), $\dot{x}_2 = u_2^*$ (s_j are the Poisson variables, the indices being in agreement with [22]) in the case of uniaxial reorientation.

The choice of the auxiliary control functions u_i^* is not unique and is governed by certain requirements concerned with the optimality of the desired control rules u_i in the original non-linear problem. Therefore u_i^* are measurable parameters (in accordance with the above requirements) in the invariable structure of u_i. In the case of interference υ_i we shall deal with the parameters u_i^* in the invariable structure of the desired control functions u_i starting from the solution of the corresponding time-minimax game problems for the auxiliary linear conflict-control systems. These are systems of the form $\ddot{\lambda}_i = u_i^* + \upsilon_i^*$ for triaxial reorientation or the corresponding form for uniaxial reorientation. Here u_i^* is the "auxiliary interference" arising in the auxiliary linear systems during the transition from the original "perturbed" non-linear system.

Within the framework of the proposed approach it is not only possible to synthesize the equations but also obtain a constructive estimate of the minimum guaranteed reorientation time. Finding it involves a computation of the "worst" interference υ_i^*, which is characteristic of game-theoretic methods. Besides, overestimated inequalities are used both in the estimates of υ_i^*, themselves and (possibly) in verifying the constraints (1.4) on u_i^*. Even though in this case the estimate of the reorientation time is not only "cautious" but somewhat excessive, as a result one can expect an acceptable guaranteed result in the solution of the complex non-linear problem.

The proposed method is one of several possible approaches to the decomposition of non-linear control systems.

The idea of the method is close to those in [24–26], from which it differs in that in the synthesis of equations it uses non-linear transformations of variables of the type introduced in [22, 23] to solve the problem of controlling the angular motion of a solid when there is no interference. Similar methods of transforming the variables are, in particular, closely related to the methods of studying problems of stabilizing dynamical systems with respect to some of the variables (partial stabilization) [13, 27–32], which have been more frequently used in recent years. This connection is due to the auxiliary role of the stabilization problem with respect to some of the variables in the course of solving the present problem of the reorientation of a body. From the viewpoint of the general analysis of stabilization and stability problems with respect to some variables (considered, respectively, in a finite or infinite time interval) this was pointed out in [13, 32]. Apart from the above-mentioned problems, an approach of this kind is also related to the methods of exact linearization of non-linear control systems [33–36] and vector Lyapunov functions applied to non-local control problems [37, 38], which have been developed in recent years.

The proposed approach is discussed in more detail below.

3. THE STRUCTURE OF GAME STRATEGIES

In accordance with the scheme outlined above we shall determine the structural form of the original game strategies starting from the solution of Problems 1 and 2 when there is no interference ($\upsilon_i \equiv 0$).

To do this we consider control rules u_i of the type [13, 22, 23]

$$u_i = \frac{1}{\lambda_0} f_i^{(0)}(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}^*) \tag{3.1}$$

where $\mathbf{u}^*$ is the auxiliary control vector formed by u_i^*. For a certain choice of $f_i^{(0)}$ one can separate the auxiliary linear control system

$$\ddot{\lambda}_i = u_i^* \tag{3.2}$$

from the closed non-linear system (1.1)–(1.3), (3.1).

System (3.2) consists of three simpler independent linear subsystems. The auxiliary control functions u_i^* in (3.2) are chosen depending on the aims of the control in the original non-linear problems 1 and 2.

The construction (3.1) is just one of possible constructions

$$u_i = \frac{1}{\lambda_j} f_i^{(j)}(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}^*), \quad j = (0,1,2,3) \tag{3.3}$$

which enable one to select auxiliary linear control systems of type (3.2) from the closed systems (1.1)–(1.3), (3.3) for a specific choice of $f_i^{(j)}$, The set of indices in (3.2) depends on the index of λ in the denominator in (3.3). The index $j = 0$ corresponds to $i = 1, 2, 3$, $j = 1 \to i = 0, 2, 3$, etc.

Along with the independent constructions (3.3), "combined" constructions are also possible, which use the independent constructions (3.3) successively. A similar "arsenal" of techniques based on (3.3) enables one, in principle, to solve the problems of controlling the angular motion of a solid for any boundary conditions [13, 23]. However, when choosing some optimal solutions it is necessary not only to optimize the independent constructions (3.3) and (when necessary) the rules for combining them, but possibly also to reduce (or damp) the angular velocity of the body. Here, as in other approaches, it is difficult to choose the precise (from the mathematical point of view) optimum solutions. However, in the iterative regime widely used in the modern applied theory of automatic control [39] a solution acceptable in practice can be constructed in real time using the proposed construction. In particular, on the basis of the solution of the corresponding linear time-optimal control problems for systems of type (3.2) one can establish time-suboptimal control rules (see [13]).

The point (and a definite advantage) of such an approach is that the original control problems 1 and 2 for the essentially non-linear system (1.1)–(1.3) can be solved on the basis of the solution of the corresponding control problems for simpler linear systems of type (3.2). This enables us to linearize the original non-linear problem without loss of control quality due to simplification.

Indeed, the variables λ_i in the auxiliary linear system (3.2) define precisely the behaviour of the same variables in the original system (1.1)–(1.3), (3.1). The remaining variables x_i and λ_0 in (1.1)–(1.3), (3.1) are connected with them by the relation

$$x_1 = \frac{2}{\lambda_0}[(\lambda_0^2 + \lambda_1^2)\dot{\lambda}_1 + (\lambda_1\lambda_2 + \lambda_0\lambda_3)\dot{\lambda}_2 + (\lambda_1\lambda_3 - \lambda_0\lambda_2)\dot{\lambda}_3] \ (1\,2\,3) \tag{3.4}$$

As a result, the transformation of the variables λ_i, $\dot{\lambda}_i$, in the auxiliary linear system (3.2) into the final state $\lambda_i = \lambda_i^1$, $\dot{\lambda}_i = 0$ means that the solid is taken into the final position $\mathbf{x}^1 = \mathbf{0}$, $\boldsymbol{\lambda} = \boldsymbol{\lambda}^1$ as required in Problems 1 and 2.

In other words, when using the proposed approach, the given state $\mathbf{x}^1 = \mathbf{0}$, $\boldsymbol{\lambda} = \boldsymbol{\lambda}^1$ of the original non-linear system (1.1)–(1.3), (3.1) is first stabilized with respect to some of the variables, namely, λ_i.The partial stabilization problem can be solved as a stabilization problem with respect to all the state variables $\lambda_i = \lambda_i^1$, $\dot{\lambda}_i = 0$ of the auxiliary linear system (3.2). Here, the partial stabilization of the given state $\mathbf{x}^1 = \mathbf{0}$, $\boldsymbol{\lambda} = \boldsymbol{\lambda}^1$ of the original non-linear system means that the state will also be stabilized with respect to the remaining variables, i.e. x_i and λ_0.

Without loss of generality we will henceforth assume that $\boldsymbol{\lambda}^1 = (1, 0, 0, 0)$. We will also confine ourselves solely to the construction of the control rules (3.1).

A number of factors increase the complexity of the solution of the auxiliary linear control problems. First, the original constraints (1.4) for u_i must be observed. This leads to the corresponding constraints

$$|u_i^*| \leq \alpha_i^* = \text{const} > 0 \tag{3.5}$$

on the auxiliary control functions u_i^* in (3.2). The procedure of fixing the numbers α_i^* can be realized constructively, for example, by choosing the numbers α_i^* recursively with subsequent verification of (1.4) along the trajectories of the corresponding closed system. To do so, one must know the explicit form of the trajectories or their acceptable estimates. Within the framework of the proposed constructions of control rules, when the original non-linear problems 1 and 2 can be reduced to simple linear control problems, such a possibility is fully guaranteed.

Note that, in principle, the constraints on the auxiliary control functions u_i^* do not have to have the same form as the constraints (1.4) on the original control functions u_i.

Now, constraints of the form $\|\mathbf{u}^*\| \leq \alpha^*$ ($\|\mathbf{u}^*\|$ being the Euclidean norm of $\mathbf{u}^*$ in R^3) are possible in place of (3.5). Conversely, the form of (1.4) is directly connected with the possibility of realizing the resulting rules u_i in some class of engines. For example, the constraints (1.4) correspond to three pairs of "fixed" engines, while the constraints $\|\mathbf{u}\| \leq \alpha$ correspond to a "vernier" engine [40]. However, the choice of the constraints for u_i^* is determined solely by mathematical considerations, namely, the search for the best acceptable control rules $\mathbf{u} \in K$ within the framework of the proposed solution scheme. From the viewpoint of the most efficient use of the capabilities of the proposed structural scheme of the control rules u_i this question requires a separate study.

It is also necessary to take into account the constraints on the phase variables λ_i when solving the corresponding control problems for the linear system (3.2). In this case the efficiency condition $|\lambda_0| \geq \varepsilon = \text{const} > 0$, $t \in [t_0, t_1]$ for constructing the control rules (3.1) applies. (The value of ε can be refined in accordance with the given constraints (1.4) on the control functions.) Naturally, it is also necessary to take (1.3) into account. As a result, the condition

$$\Sigma(\lambda_i^2) \leq 1 - \varepsilon^2, \quad t \in [t_0, t_1] \tag{3.6}$$

for λ_i in (3.2) must be met (here and henceforth the sums are taken over i from 1 to 3).

The solution of the problem of time-optimal reorientation of the solid ($\mathbf{x}^0 = \mathbf{x}^1 = \mathbf{0}$, $t_1 \to \min$) based on (3.1) involves the solution of the corresponding time-optimization problem for the auxiliary linear system (3.2) (under the constraints (3.5), which are consistent with (1.4)). In this case, being satisfied for $t = t_0$ and $t = t_1$(in the case $\boldsymbol{\lambda}^1 = (1, 0, 0, 0)$ only for $t = t_0$), condition (3.6) must be satisfied for all $t \in [t_0, t_1]$.

Indeed, we shall analyse the corresponding optimum trajectories of (3.2), taking into account the relationships $\lambda_i(t_0) = \lambda(t_1) = 0$, which follow from (1.2) on the basis of $\mathbf{x}^0 = \mathbf{x}^1 = \mathbf{0}$. The trajectories are such that $\boldsymbol{\lambda}(t) \in [\boldsymbol{\lambda}^0, \boldsymbol{\lambda}^1]$ for $t \in [t_0, t_1]$. Therefore, using (3.6) for $t = t_0$ and $t = t_1$ (in the case $\boldsymbol{\lambda}^1 = (1, 0, 0, 0)$ only for $t = t_0$), we conclude that (3.6) is also satisfied for any $t \in [t_0, t_1]$.

When studying reorientation problems with interference in order to verify (3.6) one must analyse all possible trajectories of the corresponding "perturbed" linear systems. Such an analysis is presented in Section 4.

Using (3.1) in the case with interference υ_i we obtain the "perturbed" linear system

$$\lambda_i^{\cdot} = u_i^* + \upsilon_i^* \tag{3.7}$$

in place of (3.2).

The admissible constraints on the "auxiliary interference" υ_i^* are determined by the given conditions (1.5) for υ_i. Namely, we have the qualities $\upsilon_1^* = \frac{1}{2}(\lambda_0 \upsilon_1 A_1^{-1} + \lambda_2 \upsilon_2 A_2^{-1})$ (123), from which we obtain the estimates

$$|\upsilon_i^*(t)| \leq \beta_i^* \tag{3.8}$$

$$\beta_1^* = \tfrac{1}{2}\left[\max|\lambda_0|\beta_1 A_1^{-1} + \max|\lambda_2|\beta_3 A_3^{-1} + \max|\lambda_3|\beta_2 A_2^{-1}\right] \ (1\,2\,3)$$

by virtue of (1.5).

In (3.8) the maximum is computed for $t \in [t_0, t_1]$. When $\lambda_i^0 > 0$, and $\boldsymbol{\lambda}^1 = (1, 0, 0, 0)$, we have $|\lambda_0| \leq 1$, and $|\lambda_i| \leq \lambda_1^0$. Since the estimates (3.8) are obtained using the reinforcing inequalities, the levels β_i^* of the "auxiliary interference" υ_i^* are somewhat overestimated as compared to the real ones.

4. AUXILIARY GAME-THEORETIC CONTROL PROBLEMS

In what follows we will consider the "perturbed" linear system (3.7) as a conflict-control system. As a result, in the case with interference υ_i the construction (3.1) can be regarded as the general structural form of the control rules in Problems 1 and 2. The parameters of this form, i.e. the auxiliary control rules u_i^* can be determined by solving the corresponding linear game-theoretic control problems.

Following the above assumption $\boldsymbol{\lambda}^1 = (1, 0, 0, 0)$, (without loss of generality), we solve (in accordance with the control aims in Problems 1 and 2) the problem of reducing system (3.6) to the origin of coordinates $\lambda_i = \dot{\lambda}_i = 0$ in the shortest possible time for any admissible perturbations υ_i^*. We treat this problem as a differential game, in which one of the players (the controlling party) can use the auxiliary control functions u_i^* and attempts to minimize the time τ_i of reduction to the desired state $\lambda_i = \dot{\lambda}_i = 0$. The other player (the opponent) can use the "auxiliary perturbations" υ_i^* to increase τ_i. The admissible levels of u_i^* (the quantities α_i^* in (3.5)) must be such that neither the original constraints (1.4) for u_i nor the condition (3.6) are violated. Starting from the formulation of Problems 1 and 2 to be solved, the inequalities (1.4) and (3.6) must be verified for all admissible realizations of υ_i^*. This presents a considerable difficulty when the proposed approach is used. The question is discussed in Section 5. For the time being, we assume that the levels α_i of u_i are high enough for u_i^* to exceed υ_i^*. We assume that α_i^* take some admissible values, so that $\alpha_i^* > \beta_i^*$

The above differential game for each subsystem of (3.7) is a linear differential game with objects of one type subject to the constraints $|u_i^*| \leq \alpha_i^*$ and $|\upsilon_i^*| \leq \beta_i^* = \rho_i \alpha_i^*$, $0 < \rho_i < 1$. Its solution can be reduced [19, 24] to solving the time-optimization problem for the system

$$\ddot{\lambda}_i = (1 - \rho_i) u_i^*, \quad |u_i^*| \leq \alpha_i^* \tag{4.1}$$

The boundary conditions are the same as for (3.7). The system (4.1) can be obtained from (3.7) with "auxiliary perturbations" $\upsilon_i^* = -\rho_i u_i^*$. These "worst" υ_i^*, are the optimum control functions for the "opponent". The solution of the time-optimization problem for systems of type (4.1) has the form [24, 41]

$$\begin{aligned} u_i^*(\lambda_i, \dot{\lambda}_i) &= \begin{cases} \alpha_i^* \operatorname{sign} \psi_i^\rho(\lambda_i, \dot{\lambda}_i), & \psi_i^\rho \neq 0 \\ \alpha_i^* \operatorname{sign} \dot{\lambda}_i = -\alpha_i^* \operatorname{sign} \lambda_i, & \psi_i^\rho = 0 \end{cases} \\ \psi_i^\rho(\lambda_i, \dot{\lambda}_i) &= -\lambda_i - [2\alpha_i^*(1 - \rho_i)]^{-1} \dot{\lambda}_i |\dot{\lambda}_i| \end{aligned} \tag{4.2}$$

Here Ψ_i^ρ are the switching functions.

In the case of Problem 1 the optimum trajectories of (4.1) begin and finish on the axes $\lambda_i = 0$ (Fig. 1, where $\dot{\lambda}_i^0 > 0$ to be specific).

The time $\tau_i = \tau_i(\lambda_i, \dot{\lambda}_i)$ necessary for arriving at the origin of coordinates $\lambda_i = \dot{\lambda}_i = 0$ along the optimum trajectory of system (4.1) is defined by the equalities [24, 41]

$$\tau_i = [\alpha_i^*(1 - \rho_i)]^{-1} \{[\tfrac{1}{2}\dot{\lambda}_i^2 - \alpha_i^*(1 - \rho_i)\lambda_i \operatorname{sign} \psi_i^\rho]^{1/2} - \dot{\lambda}_i \operatorname{sign} \psi_i^\rho\}$$

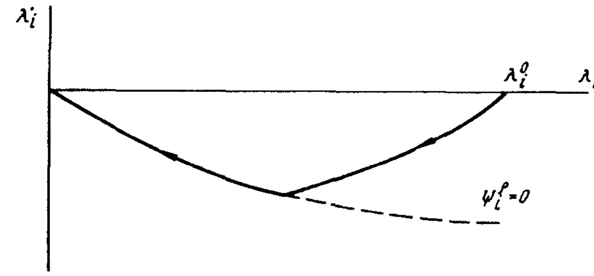

Fig. 1.

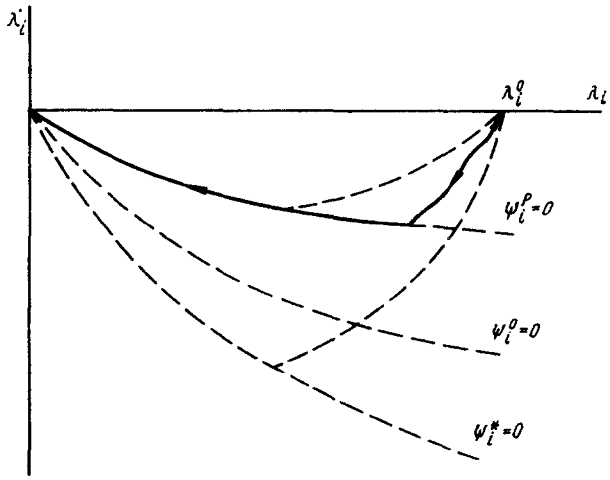

Fig. 2.

The number $\tau = \max(\tau_i)$ defines the minimum guaranteed time in the game-theoretic problem for linear system (3.7) under consideration. In the case of Problem 1, we have

$$\tau = \max(\tau_i), \quad \tau_i = 2\left\{|\lambda_i^0|\left[\alpha_i^*(1-\rho_i)\right]^{-1}\right\}^{1/2} \tag{4.3}$$

since $\dot{\lambda}_i = 0$.

Note that if υ_i^* differ from the "worst", then the time needed to move the system to the origin will not exceed τ, even though the phase trajectories of (3.7) are not optimal.

From the viewpoint of meeting the constraints (1.4) and (3.6), which are crucial in the solution of Problems 1 and 2, we will consider the characteristic features of the phase trajectories of system (3.7), (4.2) when $\upsilon_i^* \neq -\rho_i u_i^*$. We will begin by considering the case $\upsilon_i^* \equiv 0$. In this case the motion will begin on the arc of a parabola which is a trajectory of the system $\ddot{\lambda}_i = u_i^*$ for u_i^* of the form (4.2). Next, on reaching the switching curve $\psi_i^\rho = 0$, the motion will occur along this curve in the sliding regime until the desired final value $\lambda_i = \dot{\lambda}_i = 0$ is reached. During this process u_i^* takes the values $\pm\alpha_i^*$ with infinitely frequent sign changes, so that "on average" we have $u_i^* = \pm(1 = \rho_1)\alpha_i^*$ on the corresponding branches of the switching curves.

In the general case the motion occurs initially (until the switching curve is reached) between the parabolic arcs of the systems $\ddot{\lambda}_i = (1-\rho_i)u_i^*$ and $\ddot{\lambda}_i = (1+\rho_i)u_i^*$, respectively, u_i^* being of the form (4.2). (In the case of the system $\ddot{\lambda}_i = (1+\rho_i)u_i^*$ the "auxiliary interference" υ_i^* plays the role of additional control functions and has the form $\upsilon_i^* = \rho_i u_i^*$.) Then, once the switching curve $\psi_i^\rho = 0$ is reached, the system will also move along this curve in the sliding regime until the desired finite value $\lambda_i = \dot{\lambda}_i = 0$ is attained. The process of motion (in the case of Problem 1, i.e.

for $\lambda_i(t_0) = \lambda_i(t_1) = 0$) is represented by the solid line in Fig. 2, where, to be specific $\lambda_i^0 > 0$. The expressions $\psi_i^0(\psi_i^*) = 0$ can be obtained from $\psi_i^\rho = 0$ for $\rho_i = 0$ and by changing $(1-\rho_i)$ to $(1+\rho_i)$, respectively.

5. VERIFICATION OF THE PRESCRIBED CONSTRAINTS (1.4)

For the proposed construction of control rules u_i of the form (3.1), not only the switching times but also u_i themselves depend explicitly on **x** and **λ**. This makes it more difficult to verify the given constraints (1.4). Indeed, since **x** and **λ** depend on υ_i^*, the numbers α_i^* (the levels of the auxiliary control functions u_i^*) must be consistent with the conditions $|u_i(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}^*)| \leq \alpha_i$ not only for the "worst" realizations $\upsilon_i^* = -\rho_i u_i^*$, but also for any admissible realizations υ_i^*.

Thus the "worst" realizations υ_i^* must be in fact be found twice in the framework of the proposed approach. When the values of α_i^* are fixed $\upsilon_i^* = -\rho_i u_i^*$ are the "worst" realizations. This enables us, for fixed α_i^* to determine the minimum guaranteed reorientation time τ in Problems 1 and 2 as the optimum time in system (4.1). To make α_i^* consistent with the constraints (1.4) on u_i, other "worst" realizations υ_i^* must be distinguished, namely, those for which $\max|u_i|$ is attained.

Nevertheless, the constraints (1.4) can also be verified constructively by estimating $|u_i|$ on the set S of all possible states of the linear system (3.7), (4.2). To this end we refine the construction of the control rules (3.1). We differentiate both sides of each equation from the second group in (1.2), replacing x_i by their expressions from (1.1) and substituting u_i^* for $\ddot{\lambda}_i$. Solving the resulting equalities for u_i, after reduction we get

$$u_1 = 2A_1\left\{\lambda_0^{-1}\lambda_1\left[\left(\Sigma\lambda_i u_i^*\right) + \tfrac{1}{4}\Sigma x_i^2\right] + \lambda_0 u_1^* + \lambda_3 u_2^* - \lambda_2 u_3^*\right\} + (A_3 - A_2)x_2 x_3 \quad (1\,2\,3) \tag{5.1}$$

To verify (1.4) one can, in particular, use the following assertion.

Lemma. Let $\boldsymbol{\lambda}^1 = (1, 0, 0, 0)$ and $\lambda_i^0 > 0$. Assume that for all $t \in [t_0, t_0 + \tau]$ the levels α_i^* of the auxiliary control functions u_i^* in (3.7) are chosen according to the inequalities

$$2A_i(G_i + \max_{\lambda_i \in [\lambda_i^-, \lambda_i^+]} L_i) + F_i \leq \alpha_i \tag{5.2}$$

$$L_1 = \lambda_0^{-1}\lambda_1(\lambda_1\alpha_1^* + \lambda_2\alpha_2^*) + \lambda_0\alpha_1^* + \lambda_3\alpha_2^* + r_1\alpha_3^* \quad (1\,2\,3)$$

$$r_1 = \max\left(\left|\lambda_0^{-1}\lambda_1\lambda_3 - \lambda_2\right|, \left|\lambda_2 - \lambda_0^{-1}\lambda_1\lambda_3\right|\right) \quad (1\,2\,3)$$

$$G_1 = \left(\lambda_0^-\right)^{-1}\lambda_1^+\left\{\left[\left(\lambda_0^-\right)^{-1}\Sigma\left(\lambda_i^+\dot{\lambda}_i^-\right)\right]^2 + \Sigma\left(\dot{\lambda}_i^-\right)^2\right\} \quad (1\,2\,3)$$

$$F_1 = \begin{cases} 0, & \Gamma_1 = (A_3 - A_2)x_2 x_3 \operatorname{sign}(\max u_1) < 0 \\ 2|A_3 - A_2|G_1, & \Gamma_1 > 0 \quad (1\,2\,3) \end{cases}$$

Here $\lambda_i^{(+,-)}$ are the upper (respectively, lower) estimates of λ_i, $\dot{\lambda}_i^-$ are the lower estimates of $\dot{\lambda}_i$ on S , and $\lambda_0^- = [1 = \Sigma(\lambda_i^+)^2]^{1/2}$. The expressions for $\lambda_i^{(+,-)}$ and $\dot{\lambda}_i^-$ are collected in Table 1, in which $T_i = 2(1-\rho_i)^{-1}T_i^*$ and $T_i^* = \{[\alpha_i^*(1+\rho_i)]^{-1}[\lambda_i^0(1-\rho_i)]\}^{1/2}$. (Note that $T_i^* \leq \tau/2$ and $T_i \leq \tau$ for all i.)

Then the given constraints (1.4) on the control rules u_i of the form (3.1), (4.2) will be satisfied for any admissible forms of the interference υ_i ("auxiliary interference" υ_i^*).

The proof of the lemma is based on an estimate of (5.1). The relationship $\Sigma x_i^2 = 4\{[(\lambda_0)^{-1} \times \Sigma(\lambda_i\dot{\lambda}_i)]^2 + \Sigma\dot{\lambda}_i^2\}$ is used, which can be verified by direct computations using (3.4). We observe

Table 1

t	λ_i^+	λ_i^-	$\dot{\lambda}_i^-$
$[0, T_i^*]$	$\lambda_i^0 - \frac{1}{2}(1-\rho_i)\alpha_i^* t^2$	$\lambda_i^0 - \frac{1}{2}(1+\rho_i)\alpha_i^* t^2$	$-(1+\rho_i)\alpha_i^* t$
$(T_i^*, \tau/2]$	"	$\frac{1}{2}(1-\rho_i)\alpha_i^*(t-T_i)^2$	$-(1+\rho_i)\alpha_i^* T_i^*$
$(\tau/2, T_i]$	$\frac{1}{2}(1-\rho_i)\alpha_i^*(t-\tau)^2$	"	"
$(T_i, \tau]$	"	0	"

that in the procedure of finding estimates of type (5.2) no generality is lost by using the conditions $\boldsymbol{\lambda}^1 = (1, 0, 0, 0)$ and $\lambda_i^0 > 0$. For other boundary conditions estimates of the type in question can be obtained by the same scheme.

Computer simulation revealed that estimates of type (5.2) are suitable for finding $\max|u_i|$ with a view to determining the guaranteed reorientation time. On the other hand, with the same end in view it is desirable to develop an efficient computational procedure for solving the corresponding non-linear programming problem to find $\max|u_i|$ on the set S of possible states of a linear system of class (3.7), (4.2). This will make it possible to obtain a more accurate estimate of the guaranteed reorientation time.

Analysing the structure of the control rules (5.1), we also observe that the values of $\max|u_i|$ on S are sufficiently close to the values of $\max|u_i|$ on the set S^* of those possible states of the linear system (3.7), (4.2) that correspond to various combinations of "extreme" and "moderate" realizations $\upsilon_i^* = \pm\rho_i u_i^*$ and $\upsilon_i^* \equiv 0$ of υ_i^*. Moreover, the precise computation of $\max|u_i|$ on S^* does not present any difficulties.

Indeed, the structure of (5.1) is such that the growth of $\max|u_i|$ is essentially determined by the "non-synchronism" of the switching times of the auxiliary control functions u_i^*. The discussion of the set S^* of possible states of system (3.7), (4.2) takes into account (even if not completely) the "non-synchronism" of switching u_i^*. Because of this, in the subsequent computation of $\max|u_i|$ the values of $\max|u_i|$on S, where the effect of "non-synchronism" of switching u_i^* manifests itself fully, will differ only slightly from the values of S^*.

6. THE SOLUTION ALGORITHM FOR PROBLEM 1

The above discussion implies that the solution of Problem 1, concerned with the reorientation of a solid in the presence of interference, can be obtained in close form on the basis of constructions of control rules of type (3.1). Indeed, the analysis of possible trajectories of system (3.7), (4.2) carried out in Section 4 reveals that the constraints (3.6), being satisfied for $t = t_0$ and $t = t_1$ (in the case $\boldsymbol{\lambda}^1 = (1, 0, 0, 0)$ only for $t = t_0$) will also be satisfied for all possible trajectories of system (3.7), (4.2) for $t \in [t_0, t_1]$. Relationships of type (5.2) enable one to estimate constructively the guaranteed reorientation time τ.

Thus, the algorithm for solving Problem 1 consists of the following steps.

1. The preliminary choice of the construction of the control rules u_i on the basis of structural schemes of types (3.1).
2. Estimation of the "auxiliary perturbations" υ_i^*. Finding the numbers β_i^* in (3.8) from β_i and A_i.
3. Estimation of the auxiliary control functions u_i^*. The preliminary choice of α_i^*.
4. Verification of the constraints (4.1) on u_i along the trajectories of linear systems of type (3.7), (4.2) for any admissible realizations of the "auxiliary interference" υ_i^*. In doing so one can rely on relationships of type (5.2). If the inequalities (1.4) are violated, then it is necessary

to continue the search for suitable numbers α_i^*. Otherwise the guaranteed reorientation time can be determined from (4.3).

The efficiency (in the sense of the accuracy of estimating τ from the given constraints (1.4)) of the solution of Problem 1 is determined by two factors: (1) the correct choice of the structural scheme of the control rules u_i on the basis of the construction (3.3); (2) the accuracy of estimates in the verification of (1.4) for all υ_i^*. In this plan the relationships (5.2) should be regarded only as possible (admissible), entirely plausible estimates at the previous solution stage. In each specific case these estimates can be refined. It is also possible to use both optimization methods to find $\max|u_i|$.

Let us also mention the "converse" version of applying the proposed procedure of studying Problem 1. In this case, for the given boundary conditions, noise levels υ_i and values A_i an acceptable value τ of the minimum guaranteed reorientation time is prescribed. Then, knowing τ and β_i^*, one can use (4.3) to choose α_i^*. Next the values α_i are estimated for some or all admissible realizations υ_i^*. The "converse" approach can be used to estimate the capabilities of the construction of control rules (3.3) when solving Problem 1.

We summarize the above discussion in the following theorems.

Theorem 1. If the levels α_i of u_i in (1.1)–(1.3) are high enough, then for any given levels β_i of the noises υ_i the rules u_i solving Problem 1 can be constructed on the basis of relationships of type (3.1). Moreover, in Problem 1 the guaranteed result is ensured, namely, the solid is taken precisely to the prescribed state in finite time τ ($t_1 \leq t_0+\tau$) for any admissible forms of interference υ_i. The solution of the non-linear problem 1 can be reduced to solving game-theoretic problems for auxiliary linear control systems of the form (3.7). The levels β_i^* of the "auxiliary interference" υ_i^* depend on β_i and A_i and are determined by (3.8). The levels α_i^* of the auxiliary control functions u_i^* can be established in accordance with relationships of type (5.2). The value of τ depends on the boundary conditions and on α_i^* and β_i^* (which, in turn, depend on α_i, β_i and A_i) and can be determined from (4.3).

Theorem 1 defines the possible solutions of Problem 1 by constructing control rules of type (3.1). If the given constraints are imposed on u_i, then the conditions of Theorem 1 can be made more specific as follows.

Theorem 2. Let $\boldsymbol{\lambda}^1=(1, 0, 0, 0)$ and $\lambda_l^0>0$. If for the given levels α_i of u_i the levels β_i of υ_i are established from the inequalities

$$\beta_1^* = \tfrac{1}{2}(\beta_1 A_1^{-1} + \lambda_2^0\beta_3 A_3^{-1} + \lambda_3^0\beta_2 A_2^{-1}) < \alpha_1^* \quad (1\,2\,3)$$

where α_i^* satisfies conditions of type (5.2), then Problem 1 has a solution. The guaranteed reorientation time τ is then given by (4.3).

Remark. Note that it is possible to estimate the interference υ_i without using the reinforcing inequalities of type (3.8). In this case the quantities β_i^* are first taken to be arbitrary, as long as they do not violate the inequalities $\beta_i^*<\alpha_i^*$. (The quantities α_i^* are, in turn, consistent with the given condition (4.1) for υ_i for any admissible υ_i^* within the limits $|\upsilon_i^*|\leq\beta_i^*$.) The system of equations $\tfrac{1}{2}(\lambda_0\upsilon_1 A_1^{-1}+\lambda_2\upsilon_3 A_3^{-1}-\lambda_3\upsilon_2 A_2^{-1})=\upsilon_1^*\,(1\ 2\ 3)$ is then solved with respect to υ_i. Then

$$\upsilon_1 = \frac{2}{\lambda^0}\left[(\lambda_0^2+\lambda_1^2)\upsilon_1^* + (\lambda_1\lambda_2+\lambda_0\lambda_3)\upsilon_2^* + (\lambda_1\lambda_3-\lambda_0\lambda_2)\upsilon_3^*\right] \quad (1\,2\,3) \tag{6.1}$$

Using (6.1), one can estimate the "true" values of υ_i corresponding to the assigned values of β_i^*. Such estimates of υ_i are possible for each specific realization of the "auxiliary interference" υ_i^* within the limits $|\upsilon_i^*|\leq\beta_i^*$.

7. EXAMPLE 1. THE TRIAXIAL REORIENTATION OF A SPACECRAFT TAKING INTERFERENCE INTO ACCOUNT

For a spacecraft with $A_1=4\times10^4$, $A_2=8\times10^4$ and $A_3=5\times10^4$ (kg m^2) we consider Problem 1 concerned with the triaxial reorientation from the initial state $\mathbf{x}^0=\mathbf{0}$, $\boldsymbol{\lambda}^0=(0.7, 0.353, 0.434, 0.432)$ to the prescribed state $\mathbf{x}^1=\mathbf{0}$, $\boldsymbol{\lambda}^1=(1, 0, 0, 0)$.

We define the admissible limits of variation of υ_i by the inequalities

$$\tfrac{1}{2}|\lambda_0\upsilon_1A_1^{-1}+\lambda_2\upsilon_3A_3^{-1}-\lambda_3\upsilon_2A_2^{-1}|\leq\beta_1^*=10^{-3}\ (\mathrm{s}^{-2})\ (1\ 2\ 3) \tag{7.1}$$

If reinforcing inequalities of type (3.8) are then used, we find from (7.1) that

$$\tfrac{1}{2}(\beta_1A_1^{-1}+\lambda_2^0\beta_3A_3^{-1}-\lambda_3^0\beta_2A_2^{-1})\leq\beta_1^*=10^{-3}\ (\mathrm{s}^{-2})\ (1\ 2\ 3)$$

Solving the resulting system for β_i, we get $\beta_1=39.9$, $\beta_2=92.9$ and $\beta_3=57.9$ (N m). However, as will be shown, the true values of υ_i subject to the constraints (7.1) may turn out to be much larger.

We will assume that, starting from the technological manoeuvrability requirements for a spacecraft, the guaranteed reorientation time τ must not exceed $\tau=70$ (s) for the given admissible limits of variation of υ_1. We shall evaluate the processes necessary to achieve this end when using the construction of control rules u_i of the form (3.1), (4.2)

By (4.2) and (4.3), the given value τ predetermines the values α_i^* of the maximum levels of the auxiliary control functions u_i^*. We begin with the equalities $\tau=\tau_i$, which mean that the minimum guaranteed reorientation time is "levelled" with respect to each variable λ_i. As a result, we obtain the relationships

$$2\{\lambda_i^0[\alpha_i^*(1-\rho_i)]^{-1}\}^{1/2}=\tau \tag{7.2}$$

Taking into account that $\alpha_i^*(1-\rho_i)=\alpha_i^*-\beta_i^*$, where $\beta_i^*=10^{-3}$ in the case (7.1), we find from (7.2) that

$$\alpha_i^*=1.289\times10^{-3},\quad \alpha_2^*=1.354\times10^{-3},\quad \alpha_i^*=1.353\times10^{-3}\ (\mathrm{s}^{-2}) \tag{7.3}$$

Let us estimate the values $\alpha_i=\max|u_i|$ for the control rules u_i of the form (3.1), (4.2) and (7.3).

1. We will first consider the case $\upsilon_i^*=-\rho_iu_i^*$ of the "worst" υ_i^*, slowing down as much as possible the process of taking the auxiliary linear system (3.7) to the desired state $\lambda_i=\dot\lambda_i=0$. In this case $\alpha_i=\max|u_i|$ can be found along the trajectories of (4.1). The expressions necessary to carry out the computations are presented in Table 2.

Computations indicate that we have

$$\alpha_1=137.8,\quad \alpha_2=300.6,\quad \alpha_3=203.2\ (\mathrm{N\ m}) \tag{7.4}$$

along the trajectories of (4.1).

More precisely, for the "worst" interference υ_i^* the rules u_i of the form (3.1), (4.2) and (7.3) are piecewise-continuous functions with discontinuities at $t=\tau/2$, which vary in the ranges $81.0\leq|u_1|\leq137.8$, $209.7\leq|u_2|\leq300.6$, and $120.6\leq|u_3|\leq203.2$ (N m). The functions u_i are shown in Fig. 3.

Table 2

t	u_i^*	$\dot\lambda_i$	λ_i
$[0,\tau/2]$	$-\alpha_i^*$	$-(1-\rho_i)\alpha_i^*t$	$\lambda_i^0-\tfrac{1}{2}(1-\rho_i)\alpha_i^*t^2$
$(\tau/2,\tau]$	α_i^*	$(1-\rho_i)\alpha_i^*(t-\tau)$	$\tfrac{1}{2}(1-\rho_i)\alpha_i^*(t-\tau)^2$

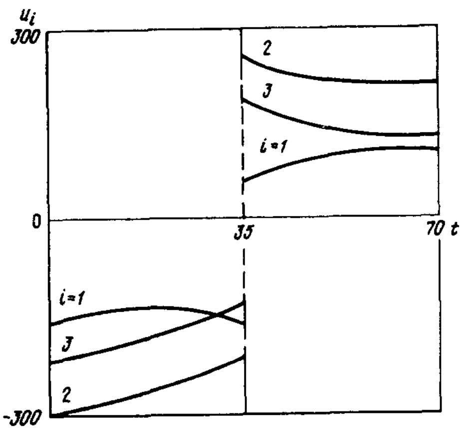

Fig. 3.

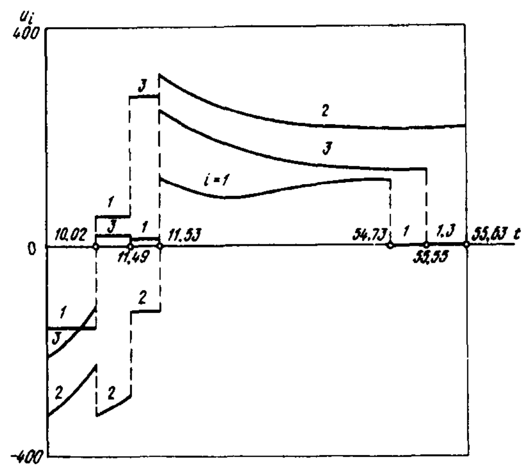

Fig. 4.

From (6.1) one can also find a program for varying the interference υ_i corresponding to the "worst" realizations υ_i^*. In the case in question υ_i are piecewise-continuous functions with discontinuities at $t=\tau/2$ varying in the ranges $80.0 \leqslant |\upsilon_1| \leqslant 105.2$, $159.1 \leqslant |\upsilon_2| \leqslant 220.4$, and $100.0 \leqslant |\upsilon_3| \leqslant 153.3$. Note that in this case the numbers $\gamma_i = |\upsilon_i| \times |u_i|^{-1}$, which characterize the relationship between υ_i and u_i, vary in the ranges $0.611 \leqslant \gamma_1 \leqslant 1.030$, $0.688 \leqslant \gamma_2 \leqslant 0.944$, and $0.634 \leqslant \gamma_3 \leqslant 0.926$, which is, on average, much higher than the "golden section" ratio ($\gamma_i > 0.618 \ldots$).

2. Next, we consider the case $\upsilon_i^* \equiv 0$ without interference. In this case the trajectories of system (3.6), (4.2), (7.3) are described in Section 4. The expressions necessary to estimate $|u_i|$ are given in Table 3. We have

$$T_i^{0*} = \{[\alpha_i^*(2-\rho_i)]^{-1}[2\lambda_i^0(1-\rho_i)]\}^{1/2}, \quad T_i^0 = (1-\rho_i)^{-1}(2-\rho_i)T_i^{0*}$$

Now, unlike the previous case, the reduction time T_i^0 will be different for each variable λ_i. We have $T_i^0 < \tau$ and $T_1^0 = 54.73$, $T_2^0 = 55.63$, and $T_3^0 = 55.55$ (s). Besides, the control rules u_i given by (3.1), (4.2), (7.2) have five switching moments $t = 10.02$, 11.49, 11.53, 54.73, 55.55 (s) and vary inside the ranges $0 \leqslant |u_i| \leqslant 137.8$, $119.2 \leqslant |u_2| \leqslant 307.7$, and $0 \leqslant |u_i| \leqslant 290.5$ (N m). The functions u_i are shown in Fig. 4, in which, for clarity, a different time scale is introduced in the sections between the points 0, 10.02, 11.49, 11.53, 54.73, 55.55, 55.63 on the t-axis. Therefore, in the case $\upsilon_i^* \equiv 0$ when there is no interference we have

$$\alpha_1 = 137.8, \quad \alpha_2 = 307.7, \quad \alpha_3 = 290.5 \text{ (N m)} \tag{7.5}$$

unlike (7.4).

To assess the performance of the construction (3.1), (4.2) in the case $\upsilon_i^* \equiv 0$ we take the switching curves $\psi_i^0 = 0$ in place of $\psi_i^\rho = 0$ in (4.2). In this construction the reorientation time $T = 70$ (s) (equal to the same value τ in the "perturbed" problem) is attained for $\alpha_1 = 53.3$, $\alpha_2 = 80.2$, and $\alpha_3 = 63.9$ (N m). However, note that for the given levels (7.1) of υ_i the construction (3.1), (4.2) with the switching curves $\psi_i^0 = 0$ cannot ensure that the body will be reoriented in finite time. Indeed, in this case $\alpha_1^* = 2.886 \times 10^{-4}$, $\alpha_2^* = 3.544 \times 10^{-4}$, and $\alpha_3^* = 3.527 \times 10^{-4}$ (s^{-2}) and ρ_i will exceed the "golden section" value for at least one i.

Table 3

t	u_i^*	$\dot\lambda_i$	λ_i
$[0, T_i^{0*}]$	$-\alpha_i^*$	$-\alpha_i^* t$	$\lambda_i^0 - \frac{1}{2}\alpha_i^* t^2$
$(T_i^{0*}, T_i^0]$	α_i^*	$(1-\rho_i)\alpha_i^*(t-T_i^0)$	$\frac{1}{2}(1-\rho_i)\alpha_i^*(t-T_i^0)^2$
$(T_i^0, \max T_i^0]$	0	0	0

Table 4

t	u_i^*	$\dot\lambda_i$	λ_i
$[0, T_i^*]$	$-\alpha_i^*$	$-(1+\rho_i)\alpha_i^* t$	$\lambda_i^0 - \frac{1}{2}(1+\rho_i)\alpha_i^* t^2$
$[T_i^*, T_i]$	α_i^*	$(1-\rho_i)\alpha_i^*(t-T_i)$	$\frac{1}{2}(1-\rho_i)\alpha_i^*(t-T_i)^2$
$[T_i, \max T_i]$	0	0	0

This means [24] that for the index i in question the auxiliary control functions u_i^* of the form (4.2) with switching curves $\psi_i^\rho = 0$ cannot transform λ_i to the state $\lambda_i = \dot\lambda_i = 0$ in finite time. At the same time, as has been shown, the control rules (3.1) and (4.2) with switching curves $\psi_i^\rho = 0$ in (4.2) do ensure the required reorientation of the body.

3. We also consider the case $\upsilon_i^* = \rho_i u_i^*$, when the "auxiliary interference" υ_i^* plays the role of the additional auxiliary control functions u_i^*. The expressions necessary for computations are presented in Table 4. In this case

$$T_i^* = \{[\alpha_i^*(1+\rho_i)]^{-1}[\lambda_i^0(1-\rho_i)]\}^{1/2}, \quad T_i = 2(1-\rho_i)^{-1}T_i^*$$

As in case 2, the reduction time T_i will be different for each variable λ_i. Moreover, $T_i < T_i^0 < \tau$ and $T_1 = 52.49$, $T_2 = 53.12$, $T_3 = 53.05$ (s). Besides, the control rules (3.1), (4.2), (7.3) have five switching moments $t = 5.88, 6.92, 6.94, 52.49, 53.05$ (s) and vary inside the ranges $0 \leqslant |u_1| \leqslant 137.8$, $82.5 \leqslant |u_2| \leqslant 338.2$, and $0 \leqslant |u_3| \leqslant 322.5$ (N m). Thus, in the case $\upsilon_i^* = \rho_i u_i^*$ we get

$$\alpha_1 = 137.8, \quad \alpha_2 = 338.2, \quad \alpha_3 = 322.5 \text{ (N m)} \tag{7.6}$$

unlike (7.4) and (7.5).

4. Now let the "auxiliary interference" υ_i^* play different roles: υ_2^*, υ_3^* are functions of the additional auxiliary control rules u_2^* and u_3^*, and $\upsilon_1^* \equiv 0$. The expressions necessary for computations can be obtained by a suitable combination of the expressions in Tables 2 and 4.

In this case the reduction times $T_2 = 53.12$ and $T_3 = 53.05$ (s) for λ_2 and λ_3 are shorter than the reduction time $T_1^0 = 54.73$ (s) for λ_1. The rules (3.1), (4.2) and (7.3) have five switching moments $t = 6.92$, 6.94, 10.02, 53.05, 53.12 (s) and vary inside the ranges $076.6 \leqslant |u_1| \leqslant 191.3$, $0 \leqslant |u_2| \leqslant 429.8$, and $0 \leqslant |u_3| \leqslant 234$ (N m). The functions u_i are shown in Fig. 5, in which, for clarity, we introduce a different time scale in the sections between the points 0, 6.92, 6.94, 10.02, 53.05, 53.12, 54.73 on the t-axis. It follows that in the case in question we have $\alpha_1 = 191.3$, $\alpha_2 = 429.8$, and $\alpha_3 = 234.8$ (N m), unlike (7.4)–(7.6).

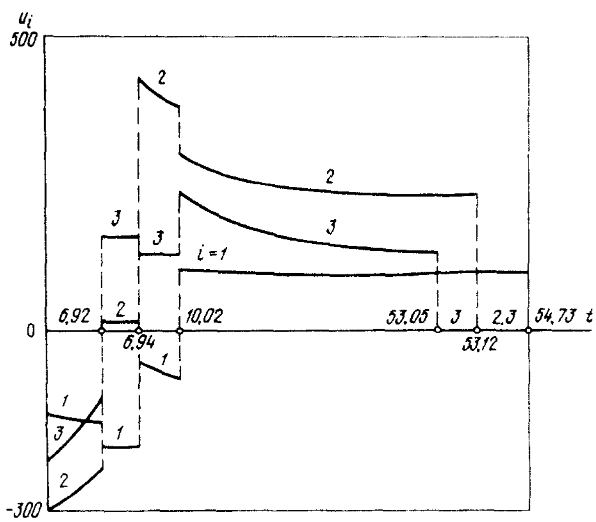

Fig. 5.

5. An analysis of cases 2–4 only indicates that the control rules u_i of the form (3.1), (4.2), (7.3) can change quite a lot when the "auxiliary interference" υ_i^* differ from the "worst" scenario $\upsilon_i^* = -\rho_i u_i^*$ (case 1). It is necessary to estimate α_i for any admissible realizations of υ_i^*. Computer modelling of estimates of type (5.2), encompassing all admissible υ_i^* at once, gives the following results: $\alpha_1 = 318.8$, $\alpha_2 = 514.5$, and $\alpha_3 = 415.9$ (N m).

Conclusions. 1. For the given boundary conditions, values of A_i, and interference levels υ_i defined in accordance with (7.1), the guaranteed reorientation time $\tau = 70$ (s) can be attained in the construction of control rules (3.1), (4.2), (7.3) for $\max\alpha_j = 514.5$ (N m). However, this is not only a "cautious", but also an overstated estimate of the admissible levels u_i.

2. For the given interference levels υ_i the "unperturbed" construction (3.1), (4.2) with switching curves $\psi_i^0 = 0$ in (4.2), in which α_i^* can be found from the computation of the reorientation time $T = 70$ (s), Problem 1 cannot be solved in a finite time.

3. The "unperturbed" construction (3.1), (4.2) with switching curves $\psi_i^0 = 0$ in (4.2), in which α_i^* can be found from the computation of the bounds $\max\alpha_j = 514.5$ (N m) (rather than the prescribed reorientation time T), guarantees the reorientation time $t = 27.64$ (s). Thus the "time expenditure" $\gamma = \tau - T$ due to the interference υ_i is equal to $\gamma = 42.36$ (s) for the given levels of u_i and υ_i and the "degree of accuracy" of the estimates.

8. AN ALGORITHM FOR SOLVING THE REORIENTATION PROBLEM WHEN THERE IS NO INTERFERENCE ($\upsilon_i \equiv 0$)

In this case the algorithm for solving Problem 1 proposed in Section 6 becomes much simpler. It involves the following steps.

1. The preliminary choice of the construction of the control rules u_i based on structural schemes of type (3.1).
2. Estimation of the auxiliary control rules u_i^*. The preliminary choice of the numbers α_i^*.
3. Verification of the constraints (1.4) and u_i along the trajectories of linear systems of type (3.2) for u_i^* of the form (4.2) with switching curves $\psi_i^0(\lambda_1, \lambda_i) = 0$ obtained from $\psi_i^\rho = 0$ for $\rho_i = 0$. If the estimates (1.4) are violated, one must continue the search for suitable numbers α_i^*. Otherwise the guaranteed reorientation time can be determined from (4.3) (for $\rho_i = 0$).

Examples of computations by this method and their comparison with the results obtained by other widely used methods are given in [13, 31]. The comparison indicates that the proposed approach is highly efficient.

It has been established by a numerical method [11] that 27.5 (s) are needed for triaxial reorientation of a spacecraft with $A_1 = 4\times10^4$, $A_2 = 8\times10^4$, and $A_3 = 5\times10^4$ (kg m^2) from $\kappa^0 = \mathbf{0}$, $\rho_0 = \psi_0 = \pi/10$, $\theta_0 = -\pi/4$ to $\kappa^1 = \mathbf{0}$, $\rho_1 = \psi_1 = \pi/6$, $\theta_1 = \pi/4$ (κ, ψ, θ are the Euler angles) subject to the conditions $|u_i| \leq 300$ (N m). (For comparison, it takes 38.5 (s) to perform a planar turn). The corresponding optimum control rules are bang-bang control rules and have one (for a plane turn) or up to four switches. Along with this, it was shown in [13, 31] that using the control rules u_i of the form (3.1), reorientation can be achieved within 29 (s). In this case u_i are piecewise-continuous and have one switching instant $t = 14.5$ (s). Comparison indicates that in the case in question the control rules u_i of the form (3.1) are suboptimal as regards speed of response.

9. ESTIMATE OF THE ADMISSIBLE LEVEL OF INTERFERENCE IN PROBLEM 1

We assume that the solution of the problem of reorienting a solid when there is no interference ($\upsilon_i \equiv 0$) (with the boundary conditions of Problem 1) has been obtained by constructing control rules u_i of type (3.1). It is assumed that the optimal speed of response problem is solved for the linear system (3.2). The constraints $|u_i| \leq \alpha_i^*$ are consistent with the given constraints (1.4). We state the following problem to estimate the efficiency of the given construction (3.1) with interference υ_i.

Problem 3. To find admissible levels β_i of υ_i, for which, as before, the construction (3.1) corresponding to the "unperturbed" case ensures that the solid will be precisely reoriented in a finite time for any admissible forms of the interference υ_i. The constraints (1.4) corresponding to the "unperturbed" case can be violated.

Problem 3 only partially characterizes the ability of the construction (3.1) corresponding to the "unperturbed" case to preserve its functional purpose. Indeed, it is not guaranteed that the original constraint on u_i will be preserved. At the same time, for a positive solution of Problem 3 it is guaranteed that the phase trajectories of system (3.7) will belong to a bounded domain of its phase plane (for u_i^* corresponding to the "unperturbed" case and for all admissible υ_i^*). This makes it possible, in principle, to give a "forecast" of α_i^*, for which not only the functional purpose of the construction (3.1) corresponding to the "unperturbed" case, but also the original constraints (1.4) are preserved. Naturally, the length of guaranteed reorientation time will increase.

The solution of Problem 3 can be constructed on the basis of the result of [24] applied to the auxiliary linear systems (3.7). Indeed [24], the presence of "auxiliary perturbations" υ_i^* in the construction of auxiliary control rules u_i^* in a system of the form (3.7) can be ignored only if the ratios ρ_i of the maximum levels υ_i^* and u_i^* do not exceed the "golden section" ratio $\rho_i = (\sqrt{(5)} - 1)/2 = 0.618\ldots$).

The following theorem can be stated on the basis of the above result and the proposed procedure for solving the reorientation problem for a solid.

Theorem 3. Let the admissible levels β_i of υ_i be determined by the conditions

$$2\beta_1^* = [\max|\lambda_0|\beta_1 A_1^{-1} + \max|\lambda_2|\beta_3 A_3^{-1} + \max|\lambda_3|\beta_2 A_2^{-1}] < (\sqrt{5} - 1)\alpha_1^* \quad (1\,2\,3)$$

(α^* are the maximum levels of the auxiliary control rules u_i^* in the "unperturbed" case). Then the problem has a solution and the given levels β_i are admissible perturbation levels in this problem.

In connection with Theorem 3 we remark that the proposed approach to the problem of the reorientation of a solid, which takes into account the effect of perturbations and is based on their game-theoretic model, guarantees a solution of Problem 3 for $\beta_i^* < \alpha_i^*$.

10. EXAMPLE 2. UNIAXIAL REORIENTATION OF A SPACECRAFT TAKING INTERFERENCE INTO ACCOUNT

For a spacecraft with the same values of A_i as in Problem 1 we consider the problem of the uniaxial reorientation of a unit vector **r** rigidly attached to it in the direction of a unit vector **s** that is stationary in an inertial space. To do this, along with Euler's equations (1.1) we consider Poisson's kinematic equation

$$\dot{s}_1 = s_2 x_3 - s_3 x_2 \quad (1\,2\,3), \quad s_1^2 + s_2^2 + s_3^2 = 1 \tag{10.1}$$

Here s_i are the projections of **s** onto the main central axes of inertia of the spacecraft. To fix our ideas, wet set $\mathbf{r} = (0, 1, 0)$. In the case in question the direction of **r** coincides with that of one of the main central axes of inertia of the spacecraft.

Consider the reorientation from the initial position $x_i = 0$, $s_1 = 0.4$, $s_3 = 0.6$ into the final position $x_i = 0$, $s_1 = s_3 = 0$. We first solve the problem setting $\upsilon_i \equiv 0$ in (1.1).

We use the control rules [23]

$$u_j = \frac{A_j}{s_2}[\varphi_j(\mathbf{x},\mathbf{s}) + \frac{s_j u_2}{A_2} \pm u_j^*] \quad (j = 1, 3), \quad u_2 = A_2 \varphi_2(\mathbf{x},\mathbf{s}) \tag{10.2}$$

The plus sign in front of the last term in the expressions for u_j ($j = 1, 3$) corresponds to $j = 1$ and the minus sign to $j=3$. The forms ϕ_j are such that the linear system

$$\dot{x}_2=0,\ \ddot{s}_1=u_3^*,\ \ddot{s}_3=u_1^* \tag{10.3}$$

can be extracted from (1.1), (10.1) and (10.2). (Note that if the indices of u_j and u_j^* in (10.2) are in agreement, this will not be so for s_j and u_j^* in (10.3).) For the linear subsystems $\ddot{s}_1=u_3^*$ and $\ddot{s}_3=u_1^*$ we solve the time-of-response optimal control problem of reduction to the origin of coordinates $s_j=\dot{s}_j=0$ ($j=1$, 3). The constraints $|u_j^*|\leq\alpha_j^*$ ($j=1$, 3) are then adopted, taking into account the given constraints $|u_i|\leq\alpha_i$.

It has been established [13] that for $\max\alpha_j\leq 170$ (Nm) the reorientation time is $T=28.3$ (s). Then $\alpha_1^*=3\times10^{-3}$ and $\alpha_3^*=2\times10^{-3}$ (s^{-2}). However, only u_1 attains the limit value, while $|u_2|\leq 14$ and $|u_3|\leq 140$ (N m). The control rules u_i are piecewise-continuous functions with discontinuities when $t=T/2$.

We consider the same reorientation problem taking into account possible effects of uncontrollable "interference" in the realization of the control rules (10.2). We assume that the "interference" arises in the course of processing the auxiliary bang-bang control functions u_j^* ($j=1$, 3). When the control structure (10.2) is considered, this leads to the condition $\upsilon_2\equiv 0$.

On changing from (1.1), (10.1) and (10.2) to (10.3), taking into account the interference υ_1 and υ_3 we obtain the "perturbed" system $\ddot{s}_j=u_j^*+\upsilon_j^*$ ($j=1$, 3). Here $\upsilon_1^*=s_2\upsilon_2A_3^{-1}$ and $\upsilon_3^*=-s_2\upsilon_1A_1^{-1}$. On the basis of Theorem 3 we conclude that if the levels β_i of υ_j ($j=1$, 3) satisfy the inequalities

$$|\beta_3A_3^{-1}|<0.618\alpha_1^*,\ |\beta_1A_1^{-1}|<0.618\alpha_3^* \tag{10.4}$$

then the control rules (10.2) guarantee that the spacecraft will be precisely reoriented in finite time for the most unfavourable action of "interference" in the control channels u_j ($j=1$, 3). Substituting into (10.4) the values α_i^* found when solving the "unperturbed" problem, we obtain estimates of admissible "interference" levels: $\beta_1<92.7$ and $\beta_3<49.4$ (N m).

11. REGARDING THE SOLUTION OF PROBLEM 2

Let us point out the properties of the solution of Problem 2. To fix our ideas let $\boldsymbol{\lambda}^1=(1, 0, 0, 0)$, $\lambda_i^0>0$, and $\dot{\lambda}_i^0>0$. This does not entail any loss of generality.

Taking the constraints (3.6) *into account*. The condition $\mathbf{x}^0=\mathbf{0}$ is not assumed in Problem 2. In this connection it is necessary, in general, to take into account the constraints (3.6), which are automatically satisfied in the case $\mathbf{x}^0=\mathbf{0}$. The constraints (3.6) will be satisfied only if the parabolic sections corresponding to the optimum trajectories of the system $\ddot{\lambda}_i=(1+\rho_i)u_i^*$ with $|u_i^*|\leq\alpha_i^*$ are "steep" enough. These parabolic sections restrict the possible trajectories (for the admissible forms of υ_i^*) of system (3.7), (4.2) until they meet the switching curves $\psi_i^\rho=0$. Such a choice of sufficiently "steep" parabolic sections imposes certain constraints on the levels α_i^* of the auxiliary control functions u_i^*. Constraints of this kind can contradict the original constraints (1.4) for u_i.

We shall determine the levels α_i^* of the auxiliary control functions u_i^* that guarantee that the conditions (3.6) are satisfied. To this end we observe that the parabolic sections of the system $\ddot{\lambda}_i=(1+\rho_i)u_i^*$, which are of interest, have the form $\lambda_i=\lambda_i^0-[2\alpha_i^*(1+\rho_i)]^{-1}[\dot{\lambda}_i^2-(\dot{\lambda}_i^0)^2]$. It follows that the points of intersection λ_i^* of these trajectories with the axes $\dot{\lambda}_i=0$ are given by

$$\lambda_i^*=\lambda_i^0+[2\alpha_i^*(1+\rho_i)]^{-1}(\dot{\lambda}_i^0)^2 \tag{11.1}$$

From (11.1), taking the inequalities $|\lambda_i(t)|\leq\lambda_i^*$ and Eqs (1.2) into account, we deduce that the desired range of values of α_i^* that guarantee that the inequalities (3.6) are satisfied can be found from the conditions

$$\Sigma\{\lambda_i^0 + [2\alpha_i^*(1+\rho_i)]^{-1}(f_i^0)^2\}^2 \leqslant 1, \tag{11.2}$$

$$2f_1^0 = x_1^0\lambda_0^0 + x_3^0\lambda_2^0 - x_2^0\lambda_3^0 \quad (1\,2\,3)$$

For any values of α_i^* that do not contradict the original constraints (1.4), one can always ensure that the conditions (11.2) are met by reducing the angular velocity of the body in advance. The problem of reducing the angular velocity can be solved on the basis of Euler's equations (1.1) alone. It is not considered in this paper. On the other hand, if the levels α_i of the original control functions u_i are high enough, conditions (11.2) can always be satisfied by choosing appropriate values of α_i^* without violating (1.4).

The non-uniqueness of the dependence of the maximum levels α_i^* of the auxiliary control functions u_i^* *on* τ. Indeed, the solution of Problem 2, even in the "unperturbed" case ($\upsilon_i \equiv \upsilon_i^* \equiv 0$) the original constraints (1.4) on the control function u_i of the form (3.1) can be satisfied for two different sets of values of α_i^*. To the two sets of α_i^* there correspond two sets of trajectories of system (3.2) leading to $\lambda_i = \dot\lambda_i = 0$. It is as if the two tendencies to increase (reduce) α_i in (1.4) by reducing (increasing) λ_0 in the denominator of (3.1) balanced one another in the above-mentioned cases. The non-uniqueness is not an obstacle to solving it. It suffices to choose the set of α_i^* that corresponds to the shortest control time. One must, however, take this property into account when solving the problem, including also the "perturbed" case.

Theorem 4. If the levels α_i of u_i are chosen in accordance with (11.2), then for any level of interference υ_i the rules u_i in Problem 2 can be stated on the basis of relationships of type (3.1). It is then ensured that Problem 1 can be solved precisely for any admissible realizations of υ_i. The solution of the non-linear Problem 2 can be reduced to solving game problems for auxiliary linear control systems of the form (3.7).

In connection with the estimate of the guaranteed reorientation time the conditions of Theorem 4 can be made more specific as in Section 6 (taking into account the above-mentioned properties).

The proposed approach to the solution of Problems 1 and 2 can be modified using other treatments of the differential game for the auxiliary linear systems of type (3.2).

12. EXAMPLE 3. TRIAXIAL REORIENTATION OF A SPACECRAFT WITH INTERFERENCE (WITH SIMULTANEOUS DAMPING OF ANGULAR VELOCITY)

For a spacecraft with the same values of A_i as in Examples 1 and 2, we consider the triaxial reorientation problem with boundary conditions $\mathbf{x}^0 = (0.0101, 0.0111, 0.0176)$, $\boldsymbol{\lambda}^0 = (0.7, 0.353, 0.434, 0.432)$ and $\mathbf{x}^1 = \mathbf{0}$, $\boldsymbol{\lambda}^1 = (1, 0, 0, 0)$. Note that the given initial value $\mathbf{x}^0$ of the angular velocity of the spacecraft corresponds to large values of the components $K_i = A_i x_i^0$ of the kinetic moment vector $\mathbf{K}$ of the spacecraft: $K_1 = 404$, $K_2 = 888$, $K_3 = 880$ (kg m^2 s^{-1}). As in Example 1, we choose the levels β_i of υ_i in accordance with the inequalities (7.1), using relationships (3.8).

We will use control rules of the form (3.1) to solve the problem. We will confine ourselves to analysing the case of the "worst" auxiliary interference υ_i^* that slow down as much as possible the process of bringing the system (3.7) to the position $\lambda_i = \dot\lambda_i = 0$ desired from the viewpoint of solving the problem in question. The expressions necessary for the computations are collected in Table 5, where

$$T_i^{**} = \{[\alpha_i^*(1-\rho_i)]^{-1}\{\dot\lambda_i^0 + [\tfrac{1}{2}(\dot\lambda_i^0)^2 + \alpha_i^*\lambda_i^0(1-\rho_i)]\}^{\frac{1}{2}}$$

$$\alpha_i^* = \beta_i^* + \varepsilon_i + (\varepsilon_i^2 + \delta_i^2)^{\frac{1}{2}}, \quad \varepsilon_i = (2T^2)^{-1}(2T\dot\lambda_i^0 + 4\lambda_i^0), \ \delta_i = T^{-1}\dot\lambda_i^0 \tag{12.1}$$

The expressions are compiled taking into account the inequalities $\lambda_i^0 > 0$ and $\dot\lambda_i^0 > 0$, which occur for the given boundary conditions, and are based on the assumption that $T = T_i$, where T_i is the reorientation

Table 5

t	u_i^*	$\dot{\lambda}_i$	λ_i
$[0,T_i^{**}]$	$-\alpha_i^*$	$\dot{\lambda}_i^0-(1-\rho_i)\alpha_i^* t$	$\lambda_i^0+\dot{\lambda}_i^0 t-½(1-\rho_i)\alpha_i^* t^2$
$[T_i^{**},T]$	α_i^*	$(1-\rho_i)\alpha_i^*(t-T)$	$½(1-\rho_i)\alpha_i^*(t-T)^2$

time for each variable λ_i. In particular, the expressions for α_i^* are obtained by solving the equations

$$T\alpha_i^*(1-\rho_i)=\dot{\lambda}_i^0+2[½(\dot{\lambda}_i^0)^2+\alpha_i^*\lambda_i^0(1-\rho_i)]^{½}$$

and using the equalities $\alpha_i^*(1-\rho_i)=\alpha_i^*-\beta_i^*$. In the course of finding the solution the roots $\alpha_i^*=\beta_i^*+\varepsilon_i-(\varepsilon_i^2+\delta_i^2)^{½}$ of the equation in question are discarded, because in this case $\alpha_i^*-\beta_i^*>0$.

Computations indicate that, for example, for $|u_i|\leq 300$ (N m) reorientation is impossible for any T in the framework of the construction of control rules (3.1) and (4.2). This is fully consistent with the first property mentioned in Section 11. At the same time, even for $\max\alpha_i=300.9$ (Nm) reorientation is possible for the case $\upsilon_i^*=-\rho_i u_i^*$ of the "worst" υ_i. In this case $T=120$ (s). The functions u_i are piecewise-continuous and have three switching moments $t=68.61$, 72.99, 73.11 (s). We emphasize that reducing as well as increasing T will result in a slow increase in α_i for a relatively long time interval. Thus, for $\max\alpha_j=302.15$ (N m) reorientation is achieved for $T=108$ and $T=130$ (s). This property is also in full agreement with the second property mentioned in Section 11.

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